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Convergence of Mayer Expansions

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The tree graph bound of Battle and Federbush is extended and used to provide a simple criterion for the convergence of (iterated) Mayer expansions. As an application estimates on the radius of convergence of the Mayer expansion for the two-dimensional Yukawa gas (nonstable interaction) are obtained.

KEY WORDS: Mayer expansion; Yukawa gas; iterated mayer expansions.

1. INTRODUCTION

This paper is largely a pedagogical exercise arising from my attemps to understand some recent work by Benfatto(1) on the two-dimensional Coulomb gas, where for the first time a direct proof of convergence of the Mayer expansion for a system, where the interaction is not classically stable, is given. "Direct" means as opposed to Ref. 2, which involves detours through cluster expansions from constructive field theory.

Mayer expansions have played an important role in the last few years.(3–5) The work by Göpfert and Mack on permanent confinement in the three-dimensional $U(1)$ gauge theory(5) largely rested on their success in improving estimates on the convergence of the Mayer expansion. In this paper I have taken ideas from Ref. 1 and combined them with estimates by Battle and Federbush(6) to arrive at a simple condition (see Theorem 2.2) for convergence of the Mayer expansion. Theorem 2.2 implies many of the results of Benfatto and Göpfert and Mack. Theorem 2.2 rests on some estimates of independent interest which are also presented in Section 2.

There are by now many accounts of the standard theory of Mayer expansions. The "classical" reference is Ref. 7. A recent review using notation cornpatible with this paper is given in Ref. 8.

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2. NOTATION AND RESULTS

Consider a system with grand canonical partition function

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^N\mu(\xi_1,\ldots,\xi_N)\, e^{-V(\xi_1,\ldots,\xi_N)} \tag{2.1}$$

where μ is a measure on a space Ω, $\mu(\Omega) < \infty$. $d\mu$ lumps together summing over species with integration over position. V is a two-body potential

$$V(\xi_1,\ldots,\xi_N) = \frac{1}{2} \sum_{i \neq j} v(\xi_i, \xi_j) \tag{2.2}$$

Assume that v is a sum of potentials (possibly different types of interactions or interactions on different scales) so that

$$\begin{aligned} v(\xi, \xi') &= \sum_{K=0}^{\infty} v^{(K)}(\xi, \xi') \\ V(\xi_1,\ldots,\xi_N) &= \sum_{K=0}^{\infty} V^{(K)}(\xi_1,\ldots,\xi_N) \end{aligned} \tag{2.3}$$

(absolute convergence, a.e. $d\mu$). Each $v^{(K)}$ is symmetric in its arguments.

Example 2.1. The Yukawa gas. There are two species, charges ± 1, with activity z which are confined to a box $\Lambda \subset R^2$:

$$\begin{aligned} Z &= \sum \frac{1}{N!} \sum_{\varepsilon_1,\ldots,\varepsilon_N = \pm 1} \int_{\Lambda^N} d^N x e^{-V(x_1,\varepsilon_1,\ldots,x_N,\varepsilon_N)} \\ &= \sum \frac{1}{N!} \int d^N\mu(\xi_1,\ldots,\xi_N)\, e^{-V} \end{aligned}$$

where $\Omega = \mathbf{R}^2 \times \{-1, 1\}$, $\xi = (x, \varepsilon) \in \Omega$, $d\mu(\xi) = (z\chi_\Lambda\, dx) \times$ (Counting measure on $\{-1, 1\}$), χ_Λ is the characteristic function of the box Λ:

$$V(\xi_1,\ldots,\xi_N) = \frac{1}{2} \beta \sum_{i \neq j} \varepsilon_i \varepsilon_j (1 - \Delta)^{-1}(x_i, x_j) \tag{2.4}$$

$\beta > 0$ is inverse temperature. The symbol $(1 - \Delta)^{-1}$ is defined by

$$(1 - \Delta)^{-1}(x, y) = \frac{1}{(2\pi)^2} \int d^2k \frac{1}{1 + k^2} e^{ik \cdot (x - y)} \tag{2.5}$$

$(1 - \Delta)^{-1}$ can be decomposed into a sum over scales:

$$= \sum_{K=0}^{\infty} u^{(K)}(x, y)$$

where for some $\gamma > 1$,

$$
\begin{aligned}
u^{(K)}(x, y) &= (\gamma^{2K} - \Delta)^{-1}(x, y) - (\gamma^{2K+2} - \Delta)^{-1}(x, y) \\
&= \frac{1}{(2\pi)^2} \int d^2k e^{ik\cdot(x-y)} \left(\frac{1}{\gamma^{2K} + k^2} - \frac{1}{\gamma^{2K+2} + k^2} \right)
\end{aligned}
\tag{2.6}
$$

To state the theorems introduce *connected parts* by

$$
(e^{-V(\xi_1,\ldots,\xi_N)})_c \equiv \sum_{\substack{G \in \text{connected} \\ \text{graphs on } \{1,2,\ldots,N\}}} \prod_{ij \in G} (e^{-V(\xi_i,\xi_j)} - 1) \qquad (=1 \text{ if } N=1) \tag{2.7}
$$

It is well known that formally

$$
\log Z = \sum_{N=1}^{\infty} \frac{1}{N!} \int d^N\mu (e^{-V})_c \tag{2.8}
$$

and this is referred to as the *Mayer expansion.* See Ref. 7 or 8 for a proof. In the existing theorems[7] on the convergence of this expansion, the *stability assumption*: that there exists a constant B such that for all $N, \xi_1,\ldots, \xi_N$,

$$
V(\xi_1,\ldots, \xi_N) \geqslant -BN \tag{2.9}
$$

plays an essential role. The advantage[4] of the iterated Mayer expansion is that this assumption can be replaced by the weaker assumption of stability at each scale: for each $K = 0, 1, 2,\ldots$ there exists $B^{(K)}$ such that for all N, $\xi_1,\ldots, \xi_N$,

$$
V^{(K)}(\xi_1,\ldots, \xi_N) \geqslant -B^{(K)}N \tag{2.10}
$$

Define

$$
\begin{aligned}
B^{(\leqslant K)} &= \sum_{i \leqslant K} B^{(i)} \\
\|v\| &= \sup_{\xi} \int d\,|\mu|\,(\xi')\,|v(\xi, \xi')|
\end{aligned}
\tag{2.11}
$$

Theorem 2.2. The Mayer expansion converges and equals $\log Z$ if

$$
\sum_{K=0}^{\infty} \|v^{(K)}\|\, e^{2B^{(\leqslant K)}} < e^{-1}
$$

Furthermore, the Nth coefficient of the Mayer expansion is bounded by

$$
\int d^N\,|\mu|\,|(e^{-V})_c| \leqslant N^{N-2} \left(\sum \|v^{(K)}\|\, e^{2B^{(\leqslant K)}} \right)^{N-1} \mu(\Omega)
$$

Corollary 2.3. For the Yukawa gas (Example 2.1) with $\beta < 4\pi$

$$\int d^N |\mu| \, |(e^{-V})_c| < 2\,|\Lambda|\, N^{N-2}\, |z|^N \left[\frac{4\beta}{2-\beta/(2\pi)}\right]^{N-1}$$

where $|\Lambda| = \text{vol}\,(\Lambda)$. Thus the Mayer expansion converges provided

$$\beta < 4\pi \qquad \text{and} \qquad 2\,|z|\,\beta\left(1-\frac{\beta}{4\pi}\right)^{-1} < e^{-1}$$

Proposition 2.4:

$$|(e^{-V(\xi_1,\dots,\xi_N)})_c| \leqslant \sum_{T,\mathbf{K}} \prod_{ij\in T} |v^{(K_{ij})}(\xi_i, \xi_j)| \times \exp\left[\sum_K B^{(K)} N^{(K)}(T, \mathbf{K})\right]$$

T is summed over all tree graphs on $\{1, 2,..., N\}$, $\mathbf{K} = (K_{ij})_{ij\in T}$ is summed over all assignments of integers to the lines ij in T and $N^{(K)}(T, \mathbf{K})$ is the number of particles which are met by lines $ij \in T$ with $K_{ij} \geqslant K$.

If the interactions $v^{(K)}$ are ordered so that the smallest interactions $v^{(K)}$ are when K is large then factors $\exp(B^{(K)}N^{(K)})$ with $N^{(K)}$ large can be matched against small $v^{(K)}$ factors. This idea, introduced in Ref. 5, is how this estimate can improve over older methods.[7,8] See the proof of Theorem 2.2 in Section 3.

Given any interaction $U = \frac{1}{2}\sum_{i\neq j} u_{ij}$ and parameters $s_{ij} \equiv s_{ji} \in [0, 1]$ define

$$U(s) = \frac{1}{2}\sum_{i\neq j} s_{ij} u_{ij} \tag{2.12}$$

If U is stable, stability is not, in general, inherited by $U(s)$. We say $U(s)$ is a *convex decoupling* of U if $U(s)$ is a convex combination of potentials of the form $\sum U(Y_i)$ where $\{Y_1,..., Y_n\}$ is a partition of $\{1, 2,..., N\}$ into disjoint subsets and $U(Y) = \frac{1}{2}\sum_{i\neq j, i,j\in Y} u_{ij}$. We refer to $U(Y)$ as *U restricted to particles in Y*. If U satisfies a stability bound, convex decouplings inherit the same bound.

Returning to V as given in (2.3) define

$$V(s) = \frac{1}{2}\sum V^{(K)}(s^{(K)}) \tag{2.13}$$

with $s^{(K)} = (s_{ij}^{(K)})$ and say that $V(s)$ is a *convex decoupling of V* if $V^{(K)}(s^{(K)})$ is a convex decoupling of $V^{(K)}$ for each K.

Given a tree graph T and an assignment $\mathbf{K}$, define for each particle i

$$K_i(T, \mathbf{K}) = \sup\{K_{lk}: lk \text{ in } T \text{ meets } i\} \tag{2.14}$$

Theorem 2.5. Let J be a positive integer. For each tree graph T on $\{1,..., N\}$ and each $\mathbf{K} = (K_{ij})_{ij \subset T}$, $0 \leqslant K_{ij} \leqslant J$, there is a probability *measure* $dP_{T,\mathbf{K}}(s)$, $s = (s_{ij}^{(\mathrm{K})})$, such that for all V given as a sum over $K \leqslant J$ as in (2.3)

(a) $(e^{-V(\xi_1,...,\xi_N)})_c = \sum_{T,K} \prod_{ij \in T} [-v^{(K_{ij})}(\xi_i, \xi_j)] \int dP_{T,K}(s)\, e^{V(\xi_1,...,\xi_N,s)}$

(b) dP_{TK} is supported on convex decouplings of V.

(c) For all s in the support of $dP_{T,\mathbf{K}}$ and each $i \in \{1,..., N\}$, $s_{ij}^{(K)} = 0$ for all j if $K > K_i(T, \mathbf{K})$.

Part (c) is an important part of this theorem. See the comment after Proposition 2.4.

3. PROOFS ASSUMING THEOREM 2.5

Proof of Proposition 2.4. Property (c) in Theorem 2.5 means that for a given pair T, $\mathbf{K}$ the particles not in the set

$$Y^{(K)} = \{i: K \leqslant K_1(T, \mathbf{K})\} \tag{3.1}$$

are decoupled from each other and all other particles, as far as $v^{(K)}$ interactions are concerned. Thus, without changing anything, we can restrict $V^{(K)}(s)$ to $Y^{(K)}$:

$$V^{(K)}(s) = \frac{1}{2} \sum_{i \neq j} s_{ij}^{(K)} v_{ij}^{(K)} \to \frac{1}{2} \sum_{\substack{i \neq j \\ i,j \in Y^{(K)}}} s_{ij}^{(K)} v_{ij}^{(K)}$$

This means that $V^{(K)}(s)$ is a convex decoupling of $V^{(K)}$ restricted to $Y^{(K)}$ and so the stability bound for $V^{(K)}$ implies

$$\begin{aligned} V^{(K)}(s) &\geqslant -B^{(K)} \,|Y^{(K)}| \\ &= -B^{(K)} N^{(K)}(T, \mathbf{K}) \end{aligned}$$

Proposition (2.4) now follows from Theorem 2.5, part (a). ∎

Proof of Theorem 2.2. Fix T, $\mathbf{K}$. Every particle in the set $Y^{(K)}$

defined in (3.1) is met by a line l in T with $K_l \geqslant K$, thus there are, at most, twice as many particles in $Y^{(K)}$ as there are such lines. This implies

$$\exp(B^{(K)}N^{(K)}(T, \mathbf{K})) \leqslant \prod_{ij \in T, K_{ij} \geqslant K} \exp(2B^{(K)})$$

We apply this inequality for $K = 0, 1, 2,\ldots$ in the right-hand side of Proposition 2.4 and find

$$|(e^{-V})_c| \leqslant \sum_{T,\mathbf{K}} \prod_{ij \in T} [|v_{ij}^{(K_{ij})}| \exp(2B^{(\leqslant K_{ij})})]$$

integrate both sides with respect to $d|\mu|$ $(\xi_1)\ldots d|\mu|$ (ξ_N) over Ω^N bounding [by the norm (2.11)] integrals over ξ_i, at the ends of extreme branches of T and working inwards to obtain

$$\int d^N\mu\, |(e^{-V})_c| \leqslant \sum_{T,\mathbf{K}} \prod_{ij \in T} [\|v^{(K_{ij})}\| \exp(2B^{(\leqslant K_{ij})})]\, \mu(\Omega)$$

Bring the $\mathbf{K}$ sum inside the product over $ij \in T$. Since all tree graphs have $N-1$ lines and there are, by Cayley's Theorem, N^{N-2} tree graphs, the bound in Theorem 2.2 is immediate. The convergence criterion follows from Sterling's theorem and the bound we have just proved applied to (2.8). ∎

Proof of Corallary 2.3. From (2.6)

$$u^{(K)}(0) = (\log \gamma)/(2\pi)$$

which implies $V^{(K)}$ obeys a stability bound with

$$B^{(K)} = \frac{1}{2}\beta \frac{\log \gamma}{2\pi}$$

Also from (2.6) and the fact that $u^{(K)}(x, y) \geqslant 0$,

$$\|v^{(K)}\| = 2\beta\gamma^{-2K}\left(1 - \frac{1}{\gamma^2}\right)$$

The 2 is from the sum over charges ± 1. The bound in Theorem 2.2 then says

$$\int d^N |\mu|\, |(e^{-V})_c| \leqslant 2\,|\Lambda|\, N^{N-2}\, |z|^N$$
$$\times \left[2\left(1 - \frac{1}{\gamma^2}\right)\right] \beta \sum_{K=0}^{\infty} \gamma^{-2K} e^{\beta(\log \gamma/2\pi)(K+1)}\Bigg]^{N-1}$$

which holds for all $\gamma > 1$ and the left-hand side is independent of γ. Take $\gamma \downarrow 1$ to obtain

$$\int d^N|\mu| \, |(e^{-V})_c| \leqslant 2\,|\Lambda|\, N^{N-2}\, |z|^N \times \left(4\beta \int dk e^{-2k + \frac{\beta}{2\pi} k}\right)^{N-1}$$

Corollary 2.3 is now immediate. ∎

4. PROOF OF THEOREM 2.5

Suppose U is given by

$$U = \frac{1}{2} \sum_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} u_{ij} \qquad (u_{ij} = u_{ji} \text{ arbitrary})$$

and, as usual, $U(s)$ is defined for $s = (s_{ij})$, $s_{ij} \in [0, 1]$ by replacing u_{ij} by $s_{ij} u_{ij}$. In Ref. 9 $[\exp(-U)]_c$ is represented as a sum over "ordered tree graphs" η. The details of this representation are not needed for this paper except to notice that it says that to each ordinary tree graph T on $\{1,..., N\}$ is associated a measure dP_T such that, for any U,

$$\text{(a)} \quad (e^{-U})_c = \sum_T \prod_{ij \in T} (-u_{ij}) \int dP_T(s)\, e^{-U(s)} \qquad (=1 \text{ if } N=1) \tag{4.1}$$

(b) dP_T is supported on convex decouplings of U.

Lemma 4.1. (Battle and Federbush[6]. Any measures (dP_T) satisfying (a) for all U are probability measures.

Proof[6]. Fix a tree graph T_0 and let U be the interaction given by

$$\begin{aligned} u_{ij} &= \varepsilon \qquad \text{if} \quad ij \in T_0 \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Then (a) reads, after substituting in the definition of $(e^{-U})_c$,

$$\prod_{ij \in T_0} (e^{-\varepsilon} - 1) = \prod_{ij \in T_0} (-\varepsilon) \int dP_{T_0} e^{-U(s)}$$

Divide both sides by ε^{N-1} and take $\varepsilon \to 0$. The result is $\int dP_{T_0} 1 = 1$. ∎

The proof of Theorem 2.5 will be by induction beginning with the interaction $V^{(J)}$ and passing to $\sum_{K=0}^{J} V^{(K)}$ via $\sum_{K=L+1}^{J} V^{(K)}$, $L=J-1$, $J-2$, $J-3,..., 0$. Equation (4.1) and Lemma 4.1 with U replaced by $V^{(J)}$ begin the induction. The inductive assumption at stage L is: to each tree graph T on $\{1, 2,..., N\}$ and $R=(K_{ij})_{ij \in T}$ with $K_{ij} \in \{L+1, L+2,..., J\}$ there is a probability measure $dP_{T,\mathbf{K}} = dP^{(L)}_{T,\mathbf{K}}$ such that for all interactions of the form $V=\sum_{L+1}^{J} V^{(K)}$, $V^{(K)}$ an arbitrary two-body interaction,

(a) $(e^{-V})_c = \sum_{T} \sum_{\mathbf{K}} \prod_{ij \in T} (-u_{ij}^{(K_{ij})}) \int dP_{T,\mathbf{K}}\, e^{-V(s)}$.

(b) $dP_{T,K}$ is supported on convex decouplings of V.

(c) For all s in the support of $dP_{T,\mathbf{K}}$ and for each

$$i \in \{1,..., N\},\ s_{ij}^{(K)} = 0 \forall j \text{ if } K > \sup\{K_{lk}, lk \in T, lk \text{ meets } i\}$$

Lemma 4.2 below will be used to accomplish the inductive step $L \to L-1$. Theorem 2.6 is proved when $L = -1$.

Given any partition Π of $\{1, 2,..., N\}$, $\Pi = \{\gamma_1,..., \gamma_N\}$, define the interaction between the sets $\gamma \in \Pi$ *induced by* W as follows:

$$\hat{W}(\Pi) = \hat{W}(\gamma_1,..., \gamma_N) = \frac{1}{2} \sum_{\substack{\gamma \cdot \gamma' \in \Pi \\ \gamma \neq \gamma'}} \hat{w}_{\gamma\gamma'} \tag{4.2}$$

where

$$\hat{w}_{\gamma\gamma'} = \sum_{i \in \gamma, j \in \gamma'} w_{ij} \tag{4.3}$$

For any $\gamma \subset \{1, 2,..., N\}$ define W *restricted to* γ by

$$W(\gamma) = \frac{1}{2} \sum_{i \neq j \in \gamma} w_{ij} \tag{4.4}$$

Lemma 4.2. Let U, W be interactions on particles $\{1, 2,..., N\}$; then

$$(e^{-U-W})_c = \sum_{\Pi} \prod_{\gamma \in \Pi} \{(e^{-U(\gamma)})_c e^{-W(\gamma)}\} (e^{-\hat{W}(\Pi)})_c$$

A proof of this Lemma is given at the end of this section. This Lemma is closely related to equations in Ref. 5 and of course Ref. 1.

The Inductive Step. In Lemma 4.2, we set

$$U = \sum_{L+1}^{J} V^{(K)}, \qquad W = V^{(L)} \tag{4.5}$$

Equation (4.1) is used to rewrite the $\{\exp[-\hat{W}(\Pi)]\}_c$ factor appearing on the right-hand side in Lemma 4.2. The inductive hypothesis (a) is used to rewrite $[\exp(-U)]_c$ factors so as to obtain

$$\begin{aligned}(\exp[-U-W])_c = \sum_{\Pi} \prod_{\gamma\in\Pi} \Bigg(& \sum_{T \text{ on } \gamma} \sum_{\mathbf{K} \text{ for } T} \prod_{ij\in T} (-v_{ij}^{(K_{ij})}) \\ & \times \int dP_{T,\mathbf{K}}(s)\, e^{-V(\gamma,s)-W(\gamma)} \Bigg) \\ & \times \sum_{\hat{T} \text{ on } \prod} \prod_{\gamma\gamma'\in\hat{T}} (-\hat{w}_{\gamma\gamma'}) \int dP_{\hat{T}}(s)\, e^{-\hat{W}(\Pi,s)} \end{aligned} \tag{4.6}$$

If some $\gamma\in\Pi$ has only one element, interpret the quantity inside ( ) as 1. The T sums unite into a sum over $T_1,..., T_n$ where T_i runs over all tree graphs on γ_i, $\Pi=\{\gamma_1,..., \gamma_n\}$. After substituting for each $\hat{w}_{\gamma\gamma'}$ using its definition (4.3), it is clear that the factors $\hat{w}_{\gamma\gamma'}$ provide all ways of linking $T_1,..., T_n$ to form a big tree graph T on $\{1, 2,..., N\}$, i.e., the sums over T_i, $\hat{T}$ and the sums inside $\hat{w}$ factors unite to form one sum over all trees T on $\{1, 2,..., N\}$. After this interpretation the sums over $\mathbf{K}_1,..., \mathbf{K}_n$ (one for each T_i) unite to one sum over all ways of assigning K_{ij} with $L+1\leqslant K_{ij}\leqslant J$ to all lines $ij\in T$ with i, j in the same $\gamma\in\Pi$. The remaining lines ij in T are from $\hat{w}$ factors and thus are $w_{ij}=v_{ij}^{(L)}$ lines. More precisely, for T fixed, $(\Pi, \mathbf{K}_1,..., \mathbf{K}_n)$ is in 1:1 correspondence with (K_{ij}), $ij\in T$, $K_{ij}=L,..., J$. For example, given T and (K^{ij}), Π is recovered by erasing all lines $ij\in T$ with $K^{ij}=L$ and then the decomposition of the resulting disconnected tree graph into connected components $T_1,..., T_n$ specifies $\Pi=\{\gamma_1,..., \gamma_n\}$ by setting $\gamma_i=$ set of particles linked by T_i, i.e., $\gamma_i\in\Pi$.

The dP probability measures in (4.6) unite into one measure $dP'_{T,\mathbf{K}}$,[2] and the interactions unite into $(U+W)(s)=\sum_{K=L}^{J} V^{(K)}(s^{(K)})$. Thus

$$(e^{-V-W})_c = \sum_{T}\sum_{\mathbf{K}} \prod_{ij\in T} (-v_{ij}^{(K_{ij})}) \int dP'_{T,\mathbf{K}}\, e^{-(V+W)(s)} \tag{4.7}$$

i.e., $dP'_{T,\mathbf{K}}$ satisfies (a) of the inductive hypotheses with L replaced by $L+1$. It is also clear from (4.6) that (b) with L replaced by $L+1$ is satisfied. To check (c) note that (Case 1) for a particle i met by any line $ij\in T$ with $K_{ij}>L$, the inductive hypothesis for dP is already the hypothesis for dP'. (Case 2) If i is only met by $ij\in T$ with $K_{ij}=L$, then in (4.6) one of the factors under the γ product equals 1 because γ has the form $\gamma=\{i\}$, and so i

[2] Some conventions are needed: if i and j lie in the same subset of the partition, the dP' distribution for $s_{ij}^{(L)}$ is a point mass at $s_{ij}^{(L)}=1$. If i and j lie in different subsets and $K>L$ the distribution $s_{ij}^{(K)}$ is a point mass at 0.

has no U interactions in the exponent. This is property (c) in this case. End of Induction and Proof of Theorem 2.6.

Proof of Lemma 4.2. By the definition of $(\ )_c$

$$
\begin{aligned}
(e^{-U-W})_c &= \sum_G \prod_{ij \in G} (e^{-u_{ij}-w_{ij}}-1) \\
&= \sum_G \prod_{ij \in G} ((e^{-u_{ij}}-1)\, e^{-w_{ij}} + (e^{w_{ij}}-1)) \\
&= \sum_{G_u, G_w} \prod_{ij \in G_u} (e^{-u_{ij}}-1)\, e^{-w_{ij}} \\
&\quad \times \prod_{k1 \in G_w} (e^{-w_{k1}}-1)
\end{aligned}
\tag{4.8}
$$

G_u, G_w are summed over all graphs on $\{1,..., N\}$ such that $G_u \cap G_w = \varnothing$ and $G_u \cup G_w$ is connected. G_u decomposes into connected components $G_1,..., G_n$ and $\{1, 2,..., N\}$ is partitioned by $\Pi = \{\gamma_1,..., \gamma_n\}$ where γ_i = set of particles linked by G_i. Thus

$$
\begin{aligned}
(e^{-U-W})_c &= \sum_{\Pi} \sum_{G_1,...,G_n \text{ on } \gamma_1,...,\gamma_n} \prod_{\substack{ij \in G_l \\ l=1,...,n}} (e^{-u_{ij}}-1)\, e^{w_{ij}} \\
&\quad \times \sum_{G_w} \prod_{kl \in G_w} (e^{-w_{kl}}-1)
\end{aligned}
\tag{4.9}
$$

The sum over G_w is also split into a sum over $P_1,..., P_n$, R, where P_i are the lines in G_w which meet only particles in γ_i, $i = 1,..., n$ and R is the rest of G_w. For γ_i, G_i fixed, the P's can be resummed using

$$
\sum_{P_i} \prod_{kl \in P_i} (e^{-w_{kl}}-1) = \exp\Bigg(-\frac{1}{2} \sum_{\substack{k \neq l \\ k,l \in \gamma_i, kl \notin G_i}} w_{kl} \Bigg)
\tag{4.10}
$$

so (4.9) becomes

$$
\begin{aligned}
(e^{-U-W})_c &= \sum_{\Pi} \sum_{G_1,...,G_n} \prod_{\substack{ij \in G_l \\ l=1,...,n}} (e^{-u_{ij}}-1) \prod_{\gamma \in \Pi} e^{-W(\gamma)} \\
&\quad \times \sum_R \prod_{kl \in R} (e^{-w_{kl}}-1)
\end{aligned}
$$

Apply definition of $[\exp(-U)]_c$:

$$
= \sum_{\Pi} \prod_{\gamma \in \Pi} (e^{-U(\gamma)})_c\, e^{-W(\gamma)} \times \sum_R \prod_{kl \in R} (e^{-w_{kl}}-1)
\tag{4.11}
$$

R is summed over all graphs which consist of lines kl with $k \in \gamma$, $l \in \gamma'$, $\gamma \neq \gamma' \in \Pi$. To each R is uniquely associated a graph $g(R)$ on $\Pi = \{\gamma_1, ..., \gamma_n\}$ with lines $\gamma\gamma'$ obtained by $kl \to \gamma\gamma'$. If $\gamma\gamma'$ occurs several times as an image of $kl \in R$, it appears in g once. Suppose $g(R)$ is fixed equal to $\{\gamma\gamma'\}$, a graph on Π with only one line, then R can be resummed using

$$\sum_{R:\, g(R) = \{\gamma\gamma'\}} \prod_{kl \in R} (e^{-w_{kl}} - 1) = (e^{\hat{w}_{\gamma\gamma'}} - 1) \tag{4.12}$$

In general $g(R)$ is a union of lines, so by taking a product of equations like the one above

$$\sum_{R:\, g(R) = g} \prod_{kl \in R} (e^{-w_{kl}} - 1) = \prod_{\gamma\gamma' \in g} (e^{-\hat{w}_{\gamma\gamma'}} - 1) \tag{4.13}$$

On substitution of this into (4.11) and using the definition of $[\exp(-\hat{W})]_c$:

$$(e^{-\hat{W}(\Pi)})_c = \sum_{\substack{g \text{ on } \Pi \\ \text{connected}}} \prod_{\gamma\gamma' \in g} (e^{-\hat{w}_{\gamma\gamma'}} - 1)$$

we obtain the conclusion of Lemma 4.2. ∎

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